

# SINGLE-PERIOD TWO-PRODUCT INVENTORY MODEL WITH SUBSTITUTION

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## ABSTRACT

In this paper we study a single-period two-product inventory model with stochastic demands, proportional revenues and costs, substitution. We focus on full downward substitution in our study.

We assume that the orders have to be placed before the demands are realized. And the problem is to decide the ordering quantity for each product. We develop a general profit maximization model for single-period two-product substitution problem, and show that it is concave and submodular. And we develop the optimization condition for the problem and rewrite the solution of the problem in a perfect form, which is similar to the solution of the newsboy problem. Then we compare the solution of this problem with the solution of newsboy problem, and proof that the profit can be improved using the substitution policy.

For the optimization solution, we study the effects of the parameters, such as the purchase cost, the penalty cost, the salvage value, the sale price. And we can get some interesting properties of the solution with respect to the parameters, and so we can know some instructions to adjust the order quantities while the parameters are changed. From the properties, we can find out the parameters that have stronger effects on the order quantities, and pay more attention on them while making the ordering decision.

## 1. INTRODUCTION

In this paper we study a single-period two-product inventory model with stochastic demands, proportional revenues and costs, substitution. A substitution is occurred whenever the demands from one product are met using stocks of another product. We focus on full downward substitution in our study. That is, demands from product 2 can be satisfied using the stocks of

product 1, but demands from product 1 can not be satisfied using the stocks of product 2. The downward substitution structure exists in real life, such as the product with higher capacity or function can satisfy the demands for the product with lower capacity or function. For example, the circuits with higher performance characters can substitute the circuits with lower performance characters in the semiconductor industry; the higher capacity memory chips can satisfy the demands for the lower capacity memory chips in the computer manufacturing factory[2]. Other examples are as follows. The steel beams with greater strength can be used to satisfy the demands for the beams with lesser strength in the steel industry[3]; the petrol with higher quality can substitute the petrol with lower quality at the petrol station.

We assume that the demands for each product are stochastic. The order, holding, penalty, and salvage costs are proportional to the quantity, and the revenue earned is also linear in the quantity sold. The orders have to be placed before the demands are realized. And the problem is to decide the ordering quantity for each product. We develop a general profit maximization model for single-period two-product substitution problem, and show that it is concave and submodular. And we develop the optimization condition for the problem and rewrite the solution of the problem in a perfect form, which is similar to the solution of the newsboy problem. Then we compare the solution of this problem with the solution of newsboy problem, and proof that the profit can be improved using the substitution policy.

For the optimization solution, we study the effects of the parameters, such as the order, penalty, salvage cost, sale price, and the variation of demands. And we can get some interesting properties of the solution with respect to the parameters. We can know how to adjust the order quantities while the parameters are changed, e.g. when the order price of product 1 decreases, the order quantity

of product 1 will increase and the order quantity of product 2 will decrease.

The rest of the paper is organized as follows. In section 2, we develop a general profit maximization model for single-period two-product substitution problem. In section 3, we develop the optimization condition for the problem and rewrite it in a perfect form, which is similar to the solution of the newsboy problem. Then we compare the optimal solution with the solution of the newsboy problem in section 4. In section 5, we show some properties of the optimal solution, which can give instructions while we are making decisions. Finally, we conclude in section 6 with a summary and the directions for the future research.

### 2. The Model

In this section, we develop a general profit maximization model for single-period two-product substitution problem.

The demand for product  $i$  is a random variable  $D_i$ , with marginal density of  $f_i(\cdot)$  and marginal distribution of  $F_i(\cdot)$ . Let  $f(\cdot)$  and  $F(\cdot)$  be the joint density and the joint distribution of demands for product 1 and product 2. For each unit of product  $i$ , the purchase cost is  $c_i$ , the selling price is  $p_i$ ,  $h_i$  is the inventory holding cost,  $\delta_i$  is the shortage penalty, and  $s_i$  is the salvage value for any surplus at the end of the period. Let  $v_i$  be the effective per unit salvage value of product  $i$ , i.e.  $v_i = s_i - h_i$ . Denote  $r_i = p_i + \delta_i$ . We make assumptions as follows.

Assumption 1:  $r_i > c_i > v_i > 0$ , for  $i = 1, 2$ .

Assumption 2:  $r_1 > r_2, c_1 > c_2, v_1 > v_2$ .

Assumption 3: if a unit of product 1 supplies the demand for product 2, the price charged is  $p_2$  (instead of  $p_1$ ).

Assumption 4:  $r_2 > v_1$ .

Assumption 1 states that each product will indeed be used to supply demand for that produce, instead of being held as inventory and exchanged for salvage value, and there is incentive for placing orders. Assumption 2 states that it is more profitable to satisfy unmet demand of product 1 than of product 2, and it is not optimal to substitute product 1 for product 2 whenever there is excess inventory of product 2. Assumption 4 states that substitution of product 1 for product 2 is profitable.

Let  $P(Q_1, Q_2)$  be the expected single period profits when the starting inventory is zero and the order quantity of product  $i$  is  $Q_i$ . It can be proved that always supply demand for product  $i$  using on-hand product  $i$  units as much as possible, and always supply the unmet demand for product 2 using the excess inventory of product 1 [4].

So the problem can be expressed as the following maximization problem:

$$\begin{aligned}
 \text{Max } P(Q_1, Q_2) = & p_1 \left[ \int_{-\infty}^{Q_1} f_1(x) dx + \int_{Q_1}^{\infty} Q_1 f_1(x) dx \right] \\
 & + p_2 \left[ \int_{-\infty}^{Q_1} \left( \int_{-\infty}^{Q_1 - Q_2 - x} y f(x, y) dy + \int_{Q_1 + Q_2 - x}^{\infty} (Q_1 + Q_2 - x) f(x, y) dy \right) dx \right. \\
 & \left. + \int_{Q_1}^{\infty} \left( \int_{-\infty}^{Q_2} y f(x, y) dy + \int_{Q_2}^{\infty} Q_2 f(x, y) dy \right) dx \right] \\
 & - c_1 Q_1 - c_2 Q_2 \\
 & + v_1 \left[ \int_{-\infty}^{Q_2} \int_{-\infty}^{Q_1 - x} f(x, y) dx dy \right. \\
 & \left. + \int_{Q_2}^{\infty} \int_{-\infty}^{Q_1 + Q_2 - y} (Q_1 + Q_2 - x - y) f(x, y) dx dy \right] \\
 & + v_2 \int_{-\infty}^{Q_2} (Q_2 - y) f_2(y) dy \\
 & - p_1 \int_{Q_1}^{\infty} (x - Q_1) f_1(x) dx \\
 & - p_2 \left[ \int_{-\infty}^{Q_1} \int_{Q_1 + Q_2 - x}^{\infty} (x + y - Q_1 - Q_2) f(x, y) dy dx \right. \\
 & \left. + \int_{Q_1}^{\infty} \int_{Q_2}^{\infty} (y - Q_2) f(x, y) dy dx \right] \tag{1}
 \end{aligned}$$

The first three lines in (1) are the revenue from supplying the demands, both directly and using substitution. The fourth line is the purchase cost. The next three lines are the net salvage value for excess inventory. The last three lines are the penalty cost for shortage.

### 3. Optimal Solution

In this section, we develop the optimal condition of the problem, and rewrite it in a perfect form that is similar to the solution of the newsboy problem.

It can be shown that  $P(Q_1, Q_2)$  is concave and submodular in  $(Q_1, Q_2)$  [1][4]. So the optimal condition can be expressed as follows:

$$\begin{cases} \frac{\partial P}{\partial Q_1} = 0 \\ \frac{\partial P}{\partial Q_2} = 0 \end{cases} \tag{2}$$

Denote that

$$G(Q_1, Q_2) = \int_{-\infty}^{Q_1} \int_{-\infty}^{Q_1 + Q_2 - x} f(x, y) dy dx \tag{3}$$

From equation (2) and (3), we can have

$$\begin{cases} F_1(Q_1^*) + \frac{r_2 - v_1}{r_1 - r_2} G(Q_1^*, Q_2^*) = \frac{r_1 - c_1}{r_1 - r_2} \\ F_2(Q_2^*) + \frac{r_2 - v_1}{r_2 - v_2} [G(Q_1^*, Q_2^*) - F(Q_1^*, Q_2^*)] = \frac{r_2 - c_2}{r_2 - v_2} \end{cases} \tag{4}$$

Where  $(Q_1^*, Q_2^*)$  are the optimal order quantities of product 1 and 2.

In equation (4), the left side is the actual service level, both directly and using substitution, and the right side is the optimal service level. It is similar to the solution of the newsboy problem.

### 4. Compare With The Newsboy Problem

In this section, we compare the order quantity of each product and the profit of this problem with that of the newsboy problem, and show the advantages of substitution policy.

Let  $(Q_1^n, Q_2^n)$  be the optimal order quantities of product 1 and 2 without substitution, i.e. the optimal solution of the newsboy problem. Let  $r_2 > v_1$  in the equation (4), we can get the optimal solution of the newsboy problem. It can be expressed as:

$$\begin{cases} F_1(Q_1^n) = \frac{r_1 - c_1}{r_1 - v_1} \\ F_2(Q_2^n) = \frac{r_2 - c_2}{r_2 - v_2} \end{cases} \quad (5)$$

We can prove that the order quantity of product 1 in the substitution scene is greater than that of the newsboy problem, and the order quantity of product 2 is less than that of the newsboy problem, that is

$$Q_1^* > Q_1^n, Q_2^* < Q_2^n \quad (6)$$

Let  $P(Q_1^n, Q_2^n)$  be the optimal expected single period profits. We can show that

$$P(Q_1^*, Q_2^*) - P(Q_1^n, Q_2^n) \geq 0 \quad (7)$$

From the optimal condition, we have

$$P(Q_1^*, Q_2^*) - P(Q_1^n, Q_2^n) \geq 0 \quad (8)$$

So, we can get

$$P(Q_1^*, Q_2^*) - P(Q_1^n, Q_2^n) \geq 0 \quad (9)$$

That is the profit can be improved by using the substitution policy.

### 5. The Properties Of Optimal Solution

In this section, we give some interesting properties of the optimal solution. They can be used to guide our ordering decision.

For the optimal solution, we can derive the first derivations of  $Q_1^*$  and  $Q_2^*$  with respect to the parameters, such as the purchase cost, the sell price, the penalty cost and the salvage value.

From equation (4), we can get

$$\begin{aligned} \frac{\partial Q_1^*}{\partial r_1} < 0, \frac{\partial Q_1^*}{\partial p_1} > 0, \frac{\partial Q_1^*}{\partial c_1} > 0, \frac{\partial Q_1^*}{\partial v_1} < 0, \\ \frac{\partial Q_2^*}{\partial r_2} > 0, \frac{\partial Q_2^*}{\partial p_2} < 0, \frac{\partial Q_2^*}{\partial c_2} > 0, \\ \frac{\partial Q_2^*}{\partial r_1} > 0, \frac{\partial Q_2^*}{\partial p_1} < 0, \frac{\partial Q_2^*}{\partial c_1} > 0, \\ \frac{\partial Q_2^*}{\partial r_2} > 0, \frac{\partial Q_2^*}{\partial p_2} < 0, \frac{\partial Q_2^*}{\partial c_2} > 0 \end{aligned} \quad (10)$$

That is said that the optimal order quantity of product 1,  $Q_1^*$ , is increasing when the sell price, the penalty cost, and the net salvage value of product 1 increase, or the

purchase cost of product 2 decreases; and  $Q_1^*$  is decreasing when the purchase cost of product 1 or the net salvage value of product 2 increase. And the optimal order quantity of product 2,  $Q_2^*$ , is increasing when the sell price, the penalty cost, and the net salvage value of product 2 increase, or the purchase cost of product 1 decreases; and  $Q_2^*$  is decreasing when the purchase cost of product 2 increases, or the sell price and the penalty cost of product 1 increase.

We also can show that the purchase cost and the salvage value have stronger effects on the order quantities, and the sell price and the penalty cost have weak effects. So when we make the order decisions, we should pay more attention on the potential changes of the purchase cost and the salvage value.

### 6. Conclusion

In this paper, we study a single-period two-product inventory model with stochastic demands, proportional revenues and costs, downward substitution, and zero leadtime of supply. We develop the optimal condition and rewrite it in a form, which is similar to the newsboy problem. Then we give some properties of the optimal solution that can guide the ordering decisions.

An obvious extension of the current work is to study the multi-period and the multi-product version of the problem. Another interesting extensions is consider the scene that the salesman has the ability to update his demand forecast as the selling season approaches, and can modify the order quantities.

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