

# Fuzzy Random Traveling Salesman Problem

Yiwen Lu, Yaodong Ni

Uncertainty Theory Laboratory, Department of Mathematical Sciences

Tsinghua University, Beijing 100084, China

lyw00@mails.tsinghua.edu.cn, nyd00@mails.tsinghua.edu.cn

**Abstract:** The travelling salesman problem is to find a shortest path from the travelling salesman's hometown, make the round of all the towns in the set, and finally go back home. This paper investigates the travelling salesman problem with fuzzy random travelling time. Three concepts are proposed: expected shortest path,  $(\alpha, \beta)$ -path and chance shortest path according to different optimal desire. Correspondingly, by using the concepts as decision criteria, three fuzzy random programming models for TSP are presented. Finally, a hybrid intelligent algorithm is designed to solve these models, and some numerical examples are provided to illustrate its effectiveness.

**Keywords:** travelling salesman problem; fuzzy random programming; fuzzy simulation; genetic algorithm.

## I. Introduction

The travelling salesman problem (TSP) is to find a shortest path from the travelling salesman's hometown, make the round of all the towns in the set, and finally go back home.

Mathematical problems related to the travelling salesman problem were treated in the 1800s by the Irish mathematician Sir William Rowan Hamilton and the British mathematician Thomas Penyngton Kirkman[3]. The term 'travelling salesman problem' was first used by J.B.Robinson[20] in 1949, which makes it clear that the TSP was already a well-known problem at that time. In 1954, G. Dantzig, R. Fulkerson, and S. Johnson[4] first solved the TSP with 49 cities, which was one of the principal events in the history of combinatorial optimization. Since then, many other researchers have made efforts to find better algorithms to solve large-scale TSP.

Traditionally, the travelling salesman problem considers the distance between two cities as a constant. However, in the real world, uncertainty always exists in travelling salesman problem. Probability theory was introduced into TSP in 1980's. In 1984, M. Mezard[18][21] studied the travelling salesman problem in the case where

the distances are random variables and found the solution under the hypothesis that the replica symmetry is not broken. This kind of TSP was called 'the random link TSP'. Lu presented three types of fuzzy models as fuzzy expected fuzzy expected value model, fuzzy  $\alpha$ -path model and most credibility model to solve fuzzy travelling salesman problem in 2005.

While in some cases, we care about the travelling time instead of the travelling distance. Considering the weather or traffic environment, the travelling time from one city to another is uncertain. Sometimes randomness and fuzziness may co-exist in traveling salesman problem in the real world. For instance, the route may change because of the traffic environment, we can assume the distance as a fuzzy variable. Everyday the speed of travel is random in different time and different weather or traffic circumstance. Therefore, we can get the shortest traveling time between two cities by the highest speed in shortest route, but we can't estimate the worst circumstance. In this case, fuzzy random variable can be introduced into optimization problems with mixed uncertainty of randomness and fuzziness. In this paper, the travelling times are assumed to be fuzzy random variables. Fuzzy random variable was first introduced by Kwakernaak[7][8]. And in the next years, Puri[19], Kruse[6], and Liu[16] developed the concept with different requirements of measurability. In this paper, we introduce the concept of fuzzy random variable initialized by Liu[16] and some relevant concepts to prepare for modelling the fuzzy random traveling salesman problem. Numerous numerical experiments have shown that the fuzzy random simulation indeed works very well. In this paper, we use a fuzzy random number to represent the travelling time between two cities, and define three concepts of shortest path in fuzzy random situation: expected shortest path(ESP),  $(\alpha, \beta)$ -path and chance shortest path(CSP). Correspondingly, by using the three concepts as decision criteria, three fuzzy random programming models are proposed.

This paper is organized as follows: In Section 2, the travelling salesman problem is described. After recalling three ranking criteria for ranking fuzzy random variables in Section 3, Section 4 builds three types of fuzzy random models as expected shortest path model,  $(\alpha, \beta)$ -path

model and chance shortest path model. Then in Section 5, some numerical experiments are given to show the effectiveness of a hybrid intelligent algorithm, which is integrated by fuzzy random simulations and genetic algorithm (GA). Furthermore, Section 6 gives some conclusions.

## II. Problem Description

The travelling salesman problem may be stated as follows: A salesman is required to visit each of some given cities once and only once, starting from one city and returning to the original place of departure. What route should he choose in order to minimize the total distance travelled?

In order to model the travelling salesman problem, we first introduce the following indices and parameters:

$i = 1, 2, \dots, n$ : cities;

$d_{ij}$ : travelling time between city  $i$  and  $j$ ;

$x_{ij}$ : decision variables, where  $x_{ij} = 0$  means that the salesman does not go from  $i$  to  $j$ , otherwise  $x_{ij} = 1$  means that the salesman goes from  $i$  to  $j$ .

Then in order to minimize the total travelling time, we have the following model:

$$\min \sum_{i \neq j} d_{ij} x_{ij} \tag{1}$$

$$s.t. \sum_{j=1}^n x_{ij} = 1, \quad i = 1, 2, \dots, n \tag{2}$$

$$\sum_{i=1}^n x_{ij} = 1, \quad j = 1, 2, \dots, n \tag{3}$$

$$\sum_{i,j \in s} x_{ij} \leq |s| - 1, \tag{4}$$

$$2 \leq |s| \leq n - 2, \quad s \subset \{1, 2, \dots, n\} \tag{4}$$

$$x_{ij} \in \{0, 1\}, \quad i, j = 1, \dots, n, \quad i \neq j. \tag{5}$$

The constraint (2) requires the salesman just goes out of city  $i$  once and the constraint (3) requires salesman just enters city  $j$  once. Hence, every city must appear once and only once in the route. The constraint (4) means that there is no circle in the route.

In the above model,  $d_{ij}$  denotes the distance between two cities and is deterministic. In the following models, we will consider  $d_{ij}$  as the travelling time between two cities and assume it to be a fuzzy random variable.  $D$  denotes the feasible set defined by the constraints (2-5).

## III. Fuzzy Random Variable

In this section, we will give some basic concepts of fuzzy random theory. The interested reader may refer to Liu[14] to see the fuzzy random theory.

We firstly recall the concepts of possibility, necessity, and credibility of a fuzzy event. Let  $\xi$  be a fuzzy variable with membership function  $\mu$ . Then the possibility, necessity, and credibility of a fuzzy event  $\{\xi \geq r\}$  can be defined by

$$\text{Pos}\{\xi \geq r\} = \sup_{u \geq r} \mu(u),$$

$$\text{Nec}\{\xi \geq r\} = 1 - \sup_{u < r} \mu(u),$$

$$\text{Cr}\{\xi \geq r\} = \frac{1}{2} (\text{Pos}\{\xi \geq r\} + \text{Nec}\{\xi \geq r\}).$$

**Definition 1** (Liu and Liu[16]) *A fuzzy random variable  $\xi$  is a function from the probability space  $(\Omega, \mathcal{A}, \text{Pr})$  to the set of fuzzy variables such that  $\text{Pos}\{\xi(\omega) \in B\}$  is a measurable function of  $\omega$  for any Borel set  $B$  of  $\mathcal{R}$ .*

### III.1 Expected Value

Next, we will introduce several concepts as ranking criteria for the preparation of modelling the traveling salesman problem. The first concept is the expected value of a fuzzy random variable. Before we give the first criterion for ranking fuzzy random variables, the concept of expected value of fuzzy variable will be given as follows:

**Definition 2** (Liu and Liu[15]) *Let  $\xi$  be a fuzzy variable. The expected value of  $\xi$  is defined by*

$$E[\xi] = \int_0^{+\infty} \text{Cr}\{\xi \geq r\} dr - \int_{-\infty}^0 \text{Cr}\{\xi \leq r\} dr$$

*provided that at least one of the above two integrals is finite.*

Based on the concept of expected value of fuzzy variable, the expected value of fuzzy random variable can be shown as:

**Definition 3** (Liu and Liu[16]) *Let  $\xi$  be a fuzzy random variable. Then its expected value is defined by*

$$E[\xi] = \int_0^{+\infty} \text{Pr}\{\omega \in \Omega \mid E[\xi(\omega)] \geq r\} dr -$$

$$\int_{-\infty}^0 \text{Pr}\{\omega \in \Omega \mid E[\xi(\omega)] \leq r\} dr$$

*provided that at least one of the above two integrals is finite.*

Let  $\xi$  and  $\eta$  be two fuzzy random variables. Liu[14] suggested that  $\xi > \eta$  if and only if  $E[\xi] > E[\eta]$ , where  $E$  is the expected value operator of fuzzy random variable.

### III.2 Chance Measure

The second concept is the chance measure of a fuzzy random event.

**Definition 4** (Liu, Gao and Liu[14]) Let  $\xi$  be a fuzzy random variable, and  $B$  a Borel set of  $\mathcal{R}$ . Then the chance of fuzzy random event  $\xi \in B$  is a function from  $(0, 1]$  to  $[0, 1]$ , defined as

$$\text{Ch}\{\xi \in B\}(\alpha) = \sup_{\text{Pr}\{A\} \geq \alpha} \inf_{\omega \in A} \text{Cr}\{\xi(\theta) \in B\}.$$

Let  $\xi$  and  $\eta$  be two fuzzy random variables. Liu[14] suggested that  $\xi > \eta$  if and only if  $\text{Ch}\{\xi \geq \bar{r}\}(\gamma) > \text{Ch}\{\eta \geq \bar{r}\}(\gamma)$  for some predetermined levels  $\bar{r}$  and  $\gamma \in (0, 1]$ .

### III.3 Critical Value

Next we will introduce two critical values as optimistic value and pessimistic value to measure fuzzy random variables.

**Definition 5** (Liu[14]) Let  $\xi$  be a fuzzy random variable, and  $\gamma, \delta \in (0, 1]$ . Then

$$\xi_{\text{sup}}(\gamma, \delta) = \sup \{r \mid \text{Ch}\{\xi \geq r\}(\gamma) \geq \delta\}$$

is called the  $(\gamma, \delta)$ -optimistic value to  $\xi$ , and

$$\xi_{\text{inf}}(\gamma, \delta) = \inf \{r \mid \text{Ch}\{\xi \leq r\}(\gamma) \geq \delta\}$$

is called the  $(\gamma, \delta)$ -pessimistic value to  $\xi$ .

Let  $\xi$  and  $\eta$  be two fuzzy random variables. Liu[14] suggested that  $\xi > \eta$  if and only if, for some predetermined confidence levels  $\gamma, \delta \in (0, 1]$ , we have  $\xi_{\text{sup}}(\gamma, \delta) > \eta_{\text{sup}}(\gamma, \delta)$ , where  $\xi_{\text{sup}}(\gamma, \delta)$  and  $\eta_{\text{sup}}(\gamma, \delta)$  are the  $(\gamma, \delta)$ -optimistic values of  $\xi$  and  $\eta$ , respectively. Similarly, Liu[14] suggested that  $\xi > \eta$  if and only if, for some predetermined confidence levels  $\gamma, \delta \in (0, 1]$ , we have  $\xi_{\text{inf}}(\gamma, \delta) > \eta_{\text{inf}}(\gamma, \delta)$ , where  $\xi_{\text{inf}}(\gamma, \delta)$  and  $\eta_{\text{inf}}(\gamma, \delta)$  are the  $(\gamma, \delta)$ -pessimistic values of  $\xi$  and  $\eta$ , respectively.

With the above concepts as different ranking criteria, we can model fuzzy random traveling salesman problem in different forms according to different types of optimization requirements.

## IV Fuzzy Random Models

### IV.1 Expected Shortest Path Model

Expected value model (EVM), which is to optimize some expected objectives with some expected constraints, is a widely used method in solving practical problems with uncertain factors. In uncertain environments, objective

functions and constraint functions always cannot be compared directly since they always involve uncertain variables. Since in fuzzy travelling salesman problem the shortest path cannot be achieved directly, it may be required to minimize the expected travelling time under the constraints.

**Definition 6** A path  $x^*$  is called the expected shortest path(ESP) if

$$E \left[ \sum_{i \neq j} d_{ij} x_{ij} \right] \geq E \left[ \sum_{i \neq j} d_{ij} x_{ij}^* \right]$$

for any path  $x$  satisfy (2.2)-(2.5).

In order to satisfy this type of request, we can build an expected valued model as:

$$\begin{aligned} \min \quad & E \left[ \sum_{i \neq j} d_{ij} x_{ij} \right] \\ \text{s.t.} \quad & \sum_{j=1}^n x_{ij} = 1, \quad i = 1, 2, \dots, n \\ & \sum_{i=1}^n x_{ij} = 1, \quad j = 1, 2, \dots, n \\ & \sum_{i,j \in s} x_{ij} \leq |s| - 1, \quad 2 \leq |s| \leq n - 2, \\ & s \subset \{1, 2, \dots, n\} \\ & x_{ij} \in \{0, 1\}, \quad i, j = 1, \dots, n, \quad i \neq j. \end{aligned}$$

### IV.2 $(\alpha, \beta)$ -Path Model

Chance-constrained programming (CCP) is a new modelling philosophy deals with uncertainty (traditionally randomness) which is initialized by Charnes[1][2]. It is applied to solve problems with the request that chance constraints should hold with at least some given confidence levels. In travelling salesman problem, a decision-maker may not want to minimize the expected travelling time, but to consider the risk. So we can establish an  $(\alpha, \beta)$ -path model to meet this type of requirement.

**Definition 7** A path  $x^*$  is called  $(\alpha, \beta)$ -path if

$$\left( \sum_{i \neq j} d_{ij} x_{ij} \right)_{\text{inf}} (\alpha, \beta) \geq \left( \sum_{i \neq j} d_{ij} x_{ij}^* \right)_{\text{inf}} (\alpha, \beta)$$

for any path  $x$  satisfy (2.2)-(2.5).

Following the idea of fuzzy random CCP, we can present the CCP models as follows:

$(\alpha, \beta)$ -path model(minimaxCCP):

$$\begin{aligned}
 \min \quad & \bar{f} \\
 \text{s.t.} \quad & \text{Ch} \left\{ \sum_{i,j} x_{ij} d_{ij} \leq \bar{f} \right\} (\alpha) \geq \beta \\
 & \sum_{j=1}^n x_{ij} = 1, \quad i = 1, 2, \dots, n \\
 & \sum_{i=1}^n x_{ij} = 1, \quad j = 1, 2, \dots, n \\
 & \sum_{i,j \in s} x_{ij} \leq |s| - 1, \quad 2 \leq |s| \leq n - 2, \\
 & s \subset \{1, 2, \dots, n\} \\
 & x_{ij} \in \{0, 1\}, \quad i, j = 1, \dots, n, \quad i \neq j.
 \end{aligned}$$

### IV.3 Chance Shortest Path Model

We have built two types of model based on the expected value and pessimistic value. Another useful tool to solve practical problems in uncertain environments is dependent-chance programming (DCP). Because many goals cannot be obtained absolutely in practice, a decision-maker may want to achieve his goal with maximal chance. Dependent-chance programming (DCP) is initialized by Liu[9] to maximize the chance functions of optimization goals. The readers may also refer to Liu[14] to see how DCP is used to solve different problems. Accordingly in fuzzy random travelling salesman problem, a decision-maker may want to maximize the chance that the total travelling time does not exceed some given time under the constraint. First, for a given travelling time  $\bar{f}$ , we obtain a fuzzy random event:  $f(x, d) = \sum_{i \neq j} d_{ij} x_{ij} \leq \bar{f}$ .

**Definition 8** A path  $x^*$  is called chance shortest path (CSP) if

$$\text{Ch} \left\{ \sum_{i \neq j} d_{ij} x_{ij} \leq \bar{f} \right\} (\alpha) \leq \text{Ch} \left\{ \sum_{i \neq j} d_{ij} x_{ij}^* \leq \bar{f} \right\} (\alpha)$$

for any given  $\bar{f}$  and  $\alpha$ .

Hence, we can build a chance shortest path model based on fuzzy random DCP:

$$\begin{aligned}
 \max \quad & \text{Ch} \left\{ \sum_{i \neq j} d_{ij} x_{ij} \leq \bar{f} \right\} (\alpha) \\
 \text{s.t.} \quad & \sum_{j=1}^n x_{ij} = 1, \quad i = 1, 2, \dots, n \\
 & \sum_{i=1}^n x_{ij} = 1, \quad j = 1, 2, \dots, n \\
 & \sum_{i,j \in s} x_{ij} \leq |s| - 1, \quad 2 \leq |s| \leq n - 2, \\
 & s \subset \{1, 2, \dots, n\} \\
 & x_{ij} \in \{0, 1\}, \quad i, j = 1, \dots, n, \quad i \neq j.
 \end{aligned}$$

## V. Numerical Experiments

In the above three models, there exist several uncertain functions with fuzzy random variables as the expected shortest path  $E \left[ \sum_{i \neq j} d_{ij} x_{ij} \right]$ , the  $(\alpha, \beta)$ -path

$\min \left\{ \bar{f} \mid \text{Ch} \left\{ \sum_{i,j} x_{ij} d_{ij} \leq \bar{f} \right\} (\alpha) \geq \beta \right\}$  and the chance  $\text{Ch} \left\{ \sum_{i,j} x_{ij} d_{ij} \leq \bar{f} \right\} (\alpha)$ . We use fuzzy random simulations introduced by Liu[13] to estimate the three fuzzy random functions.

To find a shortest path for a salesman, we need to design some heuristic algorithm. We embed the fuzzy random simulations, which are used to simulate the above three types of uncertain functions, into GA to design a hybrid intelligent algorithm. In this section, we will show the effectiveness of the hybrid intelligent algorithm by the following three numerical experiments.

Now let us consider a traveling salesman problem with 10 cities as shown in Table 1, which gives the coordinates of the cities.

Table 2 shows the matrix in which each triple denotes a fuzzy random variable.

Note that the traveling times are assumed to be fuzzy random variables, denoted by a form of triangular fuzzy variable  $(a, b, \rho)$ , where  $a$  and  $b$  are given crisp numbers and  $\rho$  is a exponential distributed random variable with the expected value is  $c$ .

Before running the hybrid intelligent algorithm, we set the parameters as follows: *pop\_size* is 50, the number of generations is 500, the number of stochastic simulation cycles is 1000, the probability of crossover  $p_c$  is 0.1, the probability of mutation  $p_m$  is 0.7, and the parameter  $a$  in the rank-based evaluation function is 0.05.

After a run of the hybrid intelligent algorithm with the parameters above to solve the expected shortest path

Table 1: Coordinates of Cities

No	City	Coordinate	No	City	Coordinate
1	Beijing	(3639,1315)	6	Handan	(3493,1696)
2	Shanghai	(4177,2244)	7	Qinhuangdao	(3904,1289)
3	Tianjin	(3712,1399)	8	Shijiazhuang	(3488,1535)
4	Baoding	(3569,1438)	9	Tangshan	(3791,1339)
5	Chengde	(3757,1187)	10	Zhangjiakou	(3506,1221)

Table 2: Fuzzy Random Matrix

0	1000,1073, exp(1146)	90,111, exp(132)	110,142, exp(174)	140,174, exp(208)	350,408, exp(466)	220,266, exp(312)	220,267, exp(314)	130,154, exp(188)	130,163, exp(196)
	0	900,964, exp(1128)	960,1010, exp(1060)	1057,1137, exp(1217)	800,876, exp(952)	910,993, exp(1076)	920,989, exp(1058)	924,984, exp(1044)	1100,1223, exp(1346)
		0	120,148, exp(176)	180,217, exp(254)	300,369, exp(438)	191,221, exp(251)	230,262, exp(294)	60,99, exp(138)	230,272, exp(314)
			0	170,314, exp(358)	220,269, exp(318)	300,367, exp(434)	100,126, exp(152)	200,243, exp(286)	190,226, exp(262)
				0	490,573, exp(656)	140,179, exp(218)	390,440, exp(490)	130,156, exp(182)	220,253, exp(286)
					0	500,578, exp(656)	120,161, exp(202)	400,465, exp(530)	410,475, exp(540)
						0	400,483, exp(566)	100,124, exp(148)	340,404, exp(468)
							0	300,361, exp(422)	250,315, exp(380)
								0	250,315, exp(380)
									0

model, an optimal path is presented as 7 – 2 – 9 – 4 – 3 – 8 – 5 – 6 – 1 – 10 and the total traveling time is 2050.

For solving the  $(\alpha, \beta)$ -path model with different confidence levels, we run the algorithm with the same parameters. The  $(\alpha, \beta)$ -path with different  $(\alpha, \beta)$  are shown as in Table 3.

Similarly, we run the hybrid intelligent algorithm for chance shortest path model. The chance shortest path with different  $\bar{f}$  are presented as in Table 4.

In order to show the effectiveness of the hybrid intelligent algorithm, we solve the  $(\alpha, \beta)$ -path model with  $(\alpha, \beta) = (0.8, 0.8)$ . We compare solutions when different parameters are taken in the algorithm ( $a = 0.05, Gen = 500, Pop.Size = 50$ ). The errors shown in Table 5 are calculated by the formula as  $(\text{actual value} - \text{optimal value}) / \text{optimal value} \times 100\%$ .

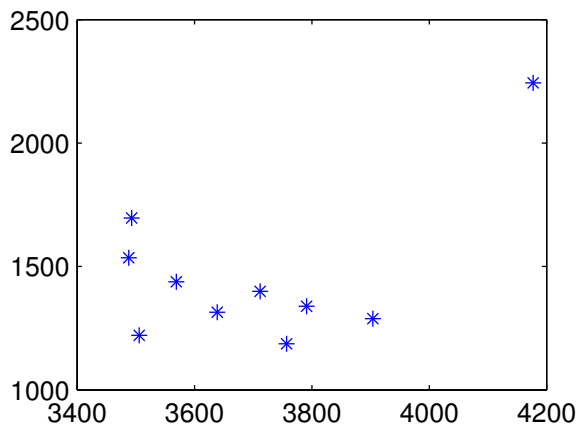


Figure 1: Positions of Cities

Table 5: Comparison Solutions of the (0.8, 0.8)-Path

$P_c$	$P_m$	(0.8, 0.8)-path	traveling time	Error(%)
0.1	0.2	10-9-5-3-1-8-4-6-7-2	2301.59	0.12%
0.2	0.3	10-8-1-3-6-5-4-9-7-2	2299.99	0.05%
0.5	0.5	3-4-6-8-5-10-7-1-9-2	2299.11	0.01%
0.7	0.8	8-9-7-5-3-1-10-4-6-2	2299.70	0.04%
0.7	0.9	9-7-4-6-8-3-5-1-10-2	2298.78	0.00%

Table 3:  $(\alpha, \beta)$ -Path

$(\alpha, \beta)$	$(\alpha, \beta)$ -Path	$(\alpha, \beta)$ -traveling time
(0.9, 0.9)	4 - 1 - 5 - 3 - 7 - 9 - 6 - 10 - 8 - 2	2423.96
(0.9, 0.8)	10 - 6 - 7 - 8 - 5 - 4 - 3 - 9 - 1 - 2	2359.68
(0.8, 0.9)	6 - 7 - 9 - 3 - 2 - 4 - 10 - 8 - 1 - 5	2341.88
(0.8, 0.8)	9 - 7 - 4 - 6 - 8 - 3 - 5 - 1 - 10 - 2	2298.78
(0.7, 0.9)	10 - 5 - 8 - 4 - 3 - 1 - 9 - 7 - 6 - 2	2256.84

Table 4: The Chance Shortest Path

$f$	the chance shortest path	chance
3000	1 - 9 - 7 - 10 - 4 - 8 - 2 - 3 - 5 - 6	0.9689
2500	3 - 9 - 7 - 5 - 6 - 1 - 10 - 8 - 4 - 2	0.9662
2300	6 - 8 - 5 - 9 - 7 - 4 - 10 - 1 - 3 - 2	0.9554
2200	10 - 1 - 6 - 7 - 8 - 4 - 3 - 5 - 9 - 2	0.8755
2100	10 - 7 - 1 - 9 - 5 - 3 - 4 - 8 - 6 - 2	0.6338
2100	10 - 7 - 1 - 9 - 5 - 3 - 4 - 8 - 6 - 2	0.6338
2000	4 - 6 - 3 - 1 - 10 - 8 - 9 - 7 - 5 - 2	0.4265

## VI. Conclusion

We introduced fuzzy random theory into traveling salesman problem and solve it with mixed uncertainty of randomness and fuzziness in this paper. After the concepts of fuzzy random theory were presented, three types of fuzzy random models as expected shortest path model,  $(\alpha, \beta)$ -path model and chance shortest path model were built. A hybrid intelligent algorithm integrated by fuzzy random simulations and genetic algorithm was designed and the results of the numerical examples were given to illustrate the effectiveness of the hybrid intelligent algorithm.

## Acknowledgments

This work was supported by Young Scholar Fund, National Natural Science Foundation of China (No.60425309), and Specialized Research Fund for the Doctoral Program of Higher Education (No.20020003009).

## References

- [1] Charnes, A. and Cooper, W.W., Chance-constrained programming, *Management Science*, Vol. 6, No. 1, 73-79, 1959.
- [2] Charnes, A., Cooper, W.W., and Thompson, G.L., Critical path analysis via chance constrained and stochastic programming, *Operational Research*, Vol. 12, 460-470, 1964.
- [3] Biggs, N.L., Lloyd, E.K., Wilson, R.J., Graph Theory, Clarendon Press, Oxford, 1976.
- [4] Dantzig, G.B., Fulkerson, D.R., Johnson, S.M., Solution of a large scale Traveling-Salesman Problem, *Opns. Res.* 2, 393-410(1954)
- [5] Kirkpatrick S, Gelatt Jr C D, Vecchi M P, Optimization by simulated annealing. *Science*, 1983, 220: 671 680
- [6] Kruse, R., and Meyer, K.D., *Statistics with Vague Data*, D.Reidel Publishing Company, Dordrecht, 1987.
- [7] Kwakernaak, H., Fuzzy random variable-I. Definitions and theorems, *Information Sciences*, Vol. 15, 1-29, 1978.
- [8] Kwakernaak, H., Fuzzy random variable-II. Algorithms and examples for the discrete case, *Information Sciences*, Vol. 17, 253-278, 1979.
- [9] Liu, B., Dependent-chance programming: A class of stochastic programming, *Computers & Mathematics with Applications*, Vol. 34, No. 12, 89-104, 1997.
- [10] Liu, B., Dependent-chance programming with fuzzy decisions, *IEEE Transactions on Fuzzy Systems*, Vol. 7, No. 3, 354-360, 1999.
- [11] Liu, B., Fuzzy random chance-constrained programming, *IEEE Transactions on Fuzzy Systems*, Vol. 9, No. 5, 713-720, 2001.
- [12] Liu, B., Fuzzy random dependent-chance programming, *IEEE Transactions on Fuzzy Systems*, Vol. 9, No. 5, 721-726, 2001.
- [13] Liu, B., *Theory and Practice of Uncertain Programming*, Heidelberg: Physica-Verlag, 2002.
- [14] Liu, B., *Uncertainty Theory: An Introduction to its Axiomatic Foundations*, Berlin: Springer-Verlag, 2004.
- [15] Liu, B., and Liu, Y.-K., Expected value of fuzzy variable and fuzzy expected value models, *IEEE Transactions on Fuzzy Systems*, Vol. 10, No. 4, 445-450, 2002.
- [16] Liu, Y.-K., and Liu, B., Fuzzy random variable: A scalar expected value operator, *Fuzzy Optimization and Decision Making*, Vol. 2, No. 2, 143-160, 2003.
- [17] Metropolis N., Rosenbluth, A., Rosenbluth M. et al, Equation of state calculations by fast computing machines. *Journal of Chemical Physics*, 1953, 21: 1087-1092
- [18] Mezard, M., Parisi, G., *J.Phys.(Paris) Lett.*, 46(1985) L-771
- [19] Puri, M.L., and Ralescu, D., Fuzzy random variables, *Journal of Mathematical Analysis and Applications*, Vol. 114, 409-422, 1986.
- [20] Robinson, J.B., On the Hamiltonian game(a traveling-salesman problem), RAND Research Memorandum RM-303.
- [21] Vannimenus, J., Mezard, M., *J.Physique Lett.* 45(1984) L-39
- [22] Zadeh, L.A., Fuzzy sets, *Information and Control*, Vol.8, 338-353, 1965